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A parachute for the degree of a polynomial in algebraically independent ones

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Abstract We give a simpler proof as well as a generalization of the main result of an article of Shestakov and Umirbaev. This latter article being the first of two that solve a long-standing conjecture about the non-tameness, or "wildness", of Nagata's automorphism. As corollaries we get interesting informations about the leading terms of polynomials forming an automorphism of $K[x_1, \ldots, x_n]$ and reprove the tameness of automorphisms of $K[x_1, x_2]$.

Keywords Degree of a polynomial · Polynomial automorphism

Mathematics Subject Classification (2000) 14R10

1 Introduction

The article [1] mentioned in the abstract gives a minoration for the degree of a polynomial in two algebraically independent ones: $G(f_1, f_2)$ (see our Corollary 2). We generalize this minoration by replacing those two by any m polynomials. Such an estimation applies in a challenging problem of affine algebraic geometry, namely finding generators of the group of polynomial automorphisms of the affine space. Indeed, given a field K, an automorphism is defined by n polynomials $f_1, \ldots, f_n$ in $K[x_1, \ldots, x_n]$ such that $K[x_1, \ldots, x_n] = K[f_1, \ldots, f_n]$. So one can write x_1 as a polynomial in the f_i's: $x_1 = G(f_1, \ldots, f_n)$. Having a minoration of the degree of $G(f_1, \ldots, f_n)$ gives thus necessary conditions that the f_i's must fulfill. In the dimension two case this condition, namely that the leading term of one of them is proportional to a power

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of the leading term of the other, suffices to build an algorithm that reduces the degree and thus to find generators.

Moreover it is also of interest to understand the behaviour of the degree of, say, $f_1 + G(f_2, \ldots, f_n)$ in order to understand the so-called "tame" subgroup of the automorphism group. Indeed the f_i's one obtains in this subgroup essentially come from chains of transformations of the type (up to reordering): $(f_1, \ldots, f_n) \rightsquigarrow (f_1 + G(f_2, \ldots, f_n), f_2, \ldots, f_n)$. In this way Shestakov and Umirbaev proved in the three-dimensional case, the non-tameness of some automorphisms (see [2]). Therefore, if our generalization of the first article of Shestakov and Umirbaev [1] doesn't really help in simplifying the second one [2], it seems however necessary (but far from being sufficient) in view of proving non-tameness results in higher dimensions.

In the present article, we give a self-contained proof of the minoration, as well as an application (Corollary 4) which constitutes a new important information about automorphisms in dimension bigger than two. Very roughly speaking we get in particular that, if $f_1, \ldots, f_n$ define an automorphism of $K[x_1, \ldots, x_n]$ and e.g. $\deg f_n \geq \deg f_i$, $\forall i$, then $\bar{f_n}$ (the leading term of f_n) is algebraic of degree at most $n-1$ over some field defined by $f_1, \ldots, f_{n-1}$. Note that, in dimension two, this gives (as already mentioned) $\bar{f_2} = \lambda \bar{f_1}^k$ and the tameness of automorphisms follows immediately (see Corollary 5).

The following notation is fixed throughout the article: K is a field of characteristic 0 and $K[x_1, \ldots, x_n]$ is the ring of polynomials in the n indeterminates $x_1, \ldots, x_n$ with coefficients in K. We endow it with the classical degree function: deg and denote by $\bar{p}$ the leading term of a polynomial $p \in K[x_1, \ldots, x_n]$. We consider m algebraically independent polynomials in $K[x_1, \ldots, x_n]$: $f_1, \ldots, f_m$ of respective degrees $d_1, \ldots, d_m$. There is also, for every polynomial $G \in K[f_1, \ldots, f_m]$, a unique one $\mathcal{G}(X_1, \ldots, X_m) \in K[X_1, \ldots, X_m]$, where $X_1, \ldots, X_m$ are new indeterminates, such that $G = \mathcal{G}(f_1, \ldots, f_m)$. By abuse of notation we will write $\frac{\partial G}{\partial f_i}$ to denote $\frac{\partial \mathcal{G}}{\partial X_i}(f_1, \ldots, f_m)$, $\forall 1 \leq i \leq m$ and $\deg_{f_i} G$ to denote $\deg_{X_i} \mathcal{G}$, the degree of $\mathcal{G}$ in X_i.

We first give an idea of the results and their proofs in the simplest non trivial case.

2 A First acquaintance in the case $m = n = 2$

To be consistent with the sequel, we regard G as a polynomial in f_2: $G = \mathcal{G}(f_1, f_2) = \Sigma g_i f_2^i$ where the g_i's are in $A := K[f_1]$. Our goal is to minorate $\deg G$ with respect to $\deg_{f_2} G$. The "generic" situation is nice since, if

$$\widehat{G} := \sum_{\deg g_i + i d_2 = \max} \bar{g}_i \bar{f}_2^i \neq 0$$

then $\deg G = \deg \widehat{G} = \max_i \{\deg g_i + i d_2\}$ and it follows that

$$\deg G \geq d_2 \deg_{f_2} G. \tag{1}$$

But it can happen that $\widehat{G} = 0$. Let us give an example:

$$\begin{cases} f_1 = x_1 + x_2^3 \ (d_1 = 3,\ \bar{f}_1 = x_2^3) \\ f_2 = x_2^2 \qquad\quad (d_2 = 2,\ \bar{f}_2 = x_2^2) \end{cases} \text{ and } G = f_1^2 + f_2 - f_2^3.$$

Then

$$\widehat{G} = \bar{f_1^2} - \bar{f_2^3} = \bar{f}_1^2 - \bar{f}_2^3 = 0 \text{ and}$$
$$G = \left(x_1 + x_2^3\right)^2 + x_2^2 - x_2^6 = x_1^2 + 2x_1x_2^3 + x_2^2$$

hence $\deg G = 4 < d_2 \deg_{f_2} G = 2 \times 3 = 6$. Whence the degree might fall with respect to the generic case. This is why we need a parachute, which comes out of the following consideration: if $\mathrm{j}(\ ,\ )$ denotes the Jacobian determinant of two polynomials with respect to x_1, x_2 it clearly fulfills

$$\frac{\partial G}{\partial f_2} \cdot \mathrm{j}(f_1, f_2) = \mathrm{j}(f_1, G) \text{ and } \deg \mathrm{j}(g_1, g_2) \leq \deg g_1 + \deg g_2 - 2,$$
$$\forall g_1, g_2 \in K[x_1, x_2].$$

Then one easily gets

$$\deg \frac{\partial G}{\partial f_2} + \deg \mathrm{j}(f_1, f_2) \leq d_1 + \deg G - 2$$
$$\deg \frac{\partial G}{\partial f_2} + d_2 - (\underbrace{d_1 + d_2 - 2 - \deg \mathrm{j}(f_1, f_2)}_{\nabla := \nabla(f_1, f_2)\ (\text{parachute of } f_1, f_2)}) \leq \deg G$$

and, by induction,

$$\deg \frac{\partial^k G}{\partial f_2^k} + k d_2 - k\nabla \leq \deg G.$$

Now we look for a "good" k, namely a k such that $\deg \frac{\partial^k G}{\partial f_2^k}$ behaves as in the generic case (1):

$$\deg \frac{\partial^k G}{\partial f_2^k} \geq d_2 \cdot \deg_{f_2} \frac{\partial^k G}{\partial f_2^k} = d_2 \cdot \deg_{f_2} G - k d_2.$$

Such a good k plugged in our inequality above would give in turn

$$d_2 \cdot \deg_{f_2} G - k\nabla \leq \deg G. \tag{2}$$

In order to find a good k, and even the "best" one, i.e. the smallest one, we consider the "symbolic leading term" of G:

$$h(X) := \sum_{\deg g_i + i\cdot d_2 = \max} \bar{g}_i X^i \in K[\bar{f}_1][X] \subset K(\bar{f}_1)[X]$$

(note that $\widehat{G} = h(\bar{f}_2)$). Now if $p = p(X) \in K(\bar{f}_1)[X]$ denotes the minimal polynomial of $\bar{f}_2$ over the field $K(\bar{f}_1)$, then, as will be shown in the proof of the Theorem 1 below, our best k is the number such that $h(X) \in (p^k)\backslash(p^{k+1})$. The proof of this fact relies on the simple remark that $\widehat{\frac{\partial G}{\partial f_m}} = h'(\bar{f}_m)$ (see (ii) in Lemma 1 below). It remains to notice that $\deg_{f_2} G = \deg_X h \geq k \deg_X p = ks_2$ ($s_2 = \deg_X p$) and inequality (2) gives

$$\deg G \geq d_2 \cdot \deg_{f_2} G - \nabla \cdot \left\lfloor \frac{\deg_{f_2} G}{s_2} \right\rfloor$$

as in the Theorem 1 below. Remark also that, by definition, $\nabla \leq d_1 + d_2 - 2$.

The general case is hardly more complicated.

3 The general case

Let us first give a definition and a property which are only formally new, and come from [1] (here replacing "two" by m is almost costless).

Definition 1 We call the *parachute* of $f_1, \dots, f_m$ and denote by $\nabla = \nabla(f_1, \dots, f_m)$ the integer

$$\nabla = d_1 + \cdots + d_m - m - \max_{1 \leq i_1, \dots, i_m \leq n} \deg \mathrm{j}_{x_{i_1}, \dots, x_{i_m}}(f_1, \dots, f_m)$$

where $\mathrm{j}_{x_{i_1}, \dots, x_{i_m}}(f_1, \dots, f_m)$ is the Jacobian determinant of $f_1, \dots, f_m$ with respect to $x_{i_1}, \dots, x_{i_m}$ that is $\mathrm{j}_{x_{i_1}, \dots, x_{i_m}}(f_1, \dots, f_m) = \det(\partial f_i / \partial x_{i_j})_{i,j}$.

For applications, such as in [1,2], it is worthwhile to notice that the parachute of $f_1, \dots, f_m$ has the following estimate:

$$0 \leq \nabla = \nabla(f_1, \dots, f_m) \leq d_1 + \cdots + d_m - m. \tag{3}$$

(actually only the right-hand majoration is useful). We however don't give the proof, which is easy, since we don't need that fact here.

Proposition 1 *For any $G \in K[f_1, \dots, f_m]$ and $\forall 1 \leq i \leq m$, one has the minoration*[1]

$$\begin{aligned} &\deg G \geq \deg \tfrac{\partial G}{\partial f_i} + d_i - \nabla \qquad \textit{and, inductively,} \\ &\deg G \geq \deg \tfrac{\partial^k G}{\partial f_i^k} + kd_i - k\nabla, \ \forall k \geq 0. \end{aligned} \tag{4}$$

[1] Hence the parachute prevents the degree from "falling too much".

Proof It is clearly sufficient to show (4) for $i = m$. Take any m integers $1 \leq i_1, \ldots, i_m \leq n$. From the definition of $\mathrm{j}_{x_{i_1},\ldots,x_{i_m}}$ it is clear that

$$\begin{aligned}\deg \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_{m-1}, G) &\leq d_1 - 1 + \cdots + d_{m-1} - 1 + \deg G - 1\\ &\leq d_1 + \cdots + d_{m-1} - m + \deg G.\end{aligned}$$

On the other hand the chain rule gives

$$\mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_{m-1}, G) = \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_{m-1}, f_m)\frac{\partial G}{\partial f_m}.$$

Hence we get

$$\deg \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_{m-1}, f_m) + \deg \frac{\partial G}{\partial f_m} \leq d_1 + \cdots + d_{m-1} - m + \deg G$$
$$\deg \frac{\partial G}{\partial f_m} + d_m - (d_1 + \cdots + d_{m-1} + d_m - m) + \deg \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_m) \leq \deg G.$$

In particular, when the maximum is realized,

$$\begin{aligned}&\deg \frac{\partial G}{\partial f_m} + d_m - (d_1 + \cdots + d_{m-1} + d_m - m)\\ &\quad + \max_{1\leq i_1,\ldots,i_m\leq n} \deg \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_m) \leq \deg G\\ &\deg \frac{\partial G}{\partial f_m} + d_m - (d_1 + \cdots + d_{m-1} + d_m - m\\ &\quad - \max_{1\leq i_1,\ldots,i_m\leq} \deg \mathrm{j}_{x_{i_1},\ldots,x_{i_m}}(f_1, \ldots, f_m)) \leq \deg G\\ &\deg \frac{\partial G}{\partial f_m} + d_m - \nabla \leq \deg G.\end{aligned}$$

In order to state our main theorem and its corollaries one needs to fix some more notation: for any subalgebra $A \subset K[x_1, \ldots, x_n]$, we denote by $\mathrm{gr}(A) := K[\bar{A}]$ the subalgebra generated by $\bar{A} = \{\bar{a} | a \in A\}$. We define s_i, $\forall 1 \leq i \leq m$, as the degree of the minimal, if any, polynomial of $\bar{f_i}$ over $\mathrm{Frac}(\mathrm{gr}(K[f_j]_{j\neq i}))$, the field of fractions of the subalgebra generated by $\overline{K[f_j]}_{j\neq i} = \overline{K[f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m]}$ and as $+\infty$ otherwise. We denote by $\lfloor \alpha \rfloor$ the integral part of a real number α and agree that $k/\infty = 0$ when $0 \leq k < \infty$.

Theorem 1 *Let G be a polynomial in $K[f_1, \ldots, f_m]$. Then the following minoration holds, $\forall 1 \leq i \leq m$,*

$$\deg G \geq d_i \cdot \deg_{f_i} G - \nabla \cdot \left\lfloor \frac{\deg_{f_i} G}{s_i} \right\rfloor.$$

Proof It is of course sufficient to prove it for $i = m$. First remark that a polynomial $G = \sum g_i f_m^i \in A[f_m]$, where $g_i \in A := K[f_1, \ldots, f_{m-1}]$, decomposes as follows (in $K[x_1, \ldots, x_n]$):

$$G = \sum_{\deg g_i + i \cdot d_m = \max} \bar{g}_i \bar{f}_m^i + rest$$

where $\deg(rest) < \max = \max_i \deg g_i + i \cdot d_m$. Consequently, G has degree strictly smaller than $\max_i \deg g_i + i \cdot d_m$ if and only if

$$\widehat{G} := \sum_{\deg g_i + i \cdot d_m = \max} \bar{g}_i \bar{f}_m^i = 0$$

so if $s_m = +\infty$, which means such an annihilation cannot occur, then the minoration in the theorem is clear. Let's assume now that $\bar{f}_m$ does have a minimal polynomial $p(\bar{f}_m) = 0$ with $p = p(X) \in F[X]$ where F is the field of fractions of $\mathrm{gr}(A)$ and X a new indeterminate (whence $s_m := \deg_X p$) and let us define

$$h(X) := \sum_{\deg g_i + i \cdot d_m = \max} \bar{g}_i X^i \in \mathrm{gr}(A)[X] \quad (\text{ hence } \widehat{G} = h(\bar{f}_m)).$$

The following easy lemma constitutes the very improvement with respect to [1]: it simplifies the proof a lot, makes it more general and even stronger in the sense that one does not need the estimate (3) anymore.

Lemma 1 (i) *If* $\deg G < \deg_{f_m} G \cdot d_m$ *then* $\widehat{G} = 0$ *or, equivalently,* $h(X) \in p(X) \cdot F[X]$.

(ii) *If* $h'(X) \neq 0$, *where* h' *is the derivative of* h, *then* $\widehat{\frac{\partial G}{\partial f_m}} = h'(\bar{f}_m)$ *and more generally, while* $h^{(k)}(X) \neq 0$, *one has* $\widehat{\frac{\partial^k G}{\partial f_m^k}} = h^{(k)}(\bar{f}_m)$.

Proof (i) If $\deg G < \deg_{f_m} G \cdot d_m$ then $\deg G < \max_i \deg g_i + i \cdot d_m$ and $\widehat{G} = 0$ as already remarked above. This means $h(\bar{f}_m) = 0$ i.e. $h(X)$ is a multiple of $p(X)$, by definition of $p(X)$.

(ii) Assume that $h' \neq 0$. One has

$$\widehat{G} = \sum_{i \in I} \bar{g}_i \bar{f}_m^i = h(\bar{f}_m) \text{ where } I := \{i \mid \deg g_i + i \cdot d_m \geq \deg g_j + j \cdot d_m \ \forall j\} \text{ and}$$

$$\widehat{\frac{\partial G}{\partial f_m}} = \sum_{i \in I'} i \bar{g}_i \bar{f}_m^{i-1} \text{ where } I' := \{i \mid \deg i g_i + (i-1) \cdot d_m \geq \deg j g_j + (j-1) \cdot d_m \ \forall j\}.$$

Now notice that $I' = I \cap \mathbb{N}^*$ when this intersection is not empty, which occurs exactly when $h' \neq 0$. It follows directly from $I' = I \cap \mathbb{N}^*$ that $h'(\bar{f}_m) = \sum_{i \in I'} i \bar{g}_i \bar{f}_m^{i-1} = \widehat{\frac{\partial G}{\partial f_m}}$.

Let now k be the maximal number such that $h = h(X) \in (p^k) := p(X)^k \cdot F[X]$. Clearly $\deg_{f_m} G \geq \deg h \geq k \cdot \deg p = ks_m$ hence $k \leq \left\lfloor \frac{\deg_{f_m} G}{s_m} \right\rfloor$. One has moreover $h^{(k)} \notin (p)$ indeed $h \in (p^k) \backslash (p^{k+1}) \Leftrightarrow h = qp^k$ with $q \notin (p) \Rightarrow h' = q'p^k + kqp^{k-1} = (q'p + kq)p^{k-1} \in (p^{k-1}) \backslash (p^k)$ (since p does not divide q nor, consequently, $q'p + kq) \Rightarrow h'' \in (p^{k-2}) \backslash (p^{k-1}) \Rightarrow \cdots \Rightarrow h^{(k)} \notin (p)$.

It follows from $h^{(k)} \notin (p(X))$, by Lemma 1:

$$\deg \frac{\partial^k G}{\partial f_m^k} \geq d_m \cdot \deg_{f_m} \frac{\partial^k G}{\partial f_m^k} = d_m \cdot (\deg_{f_m} G - k)$$

and, by property (4),

$$\begin{aligned}\deg G &\geq d_m \cdot (\deg_{f_m} G - k) + k \cdot d_m - k \cdot \nabla = d_m \cdot \deg_{f_m} G - k \cdot \nabla \\ &\geq d_m \cdot \deg_{f_m} G - \nabla \cdot \left\lfloor \frac{\deg_{f_m} G}{s_m} \right\rfloor.\end{aligned}$$

A straightforward computation gives the following

Corollary 1 *Define,* $\forall i = 1, \ldots, m$, $N_i = N_i(f_1, \ldots, f_m) := s_i d_i - \nabla$. *Let* G *be a polynomial in* $K[f_1, \ldots, f_m]$ *and,* $\forall i = 1, \ldots, m$, *let* $\deg_{f_i} G = q_i s_i + r_i$ *be the euclidean division of* $\deg_{f_i} G$ *by* s_i *if* $s_i < \infty$ *and* $\deg_{f_i} G = r_i$ *otherwise. Then the following minoration holds*

$$\deg G \geq q_i \cdot N_i + r_i d_i$$

where $q_i \cdot N_i := 0$ *if* $s_i = \infty$.

The special case $m = 2$ corresponds to the main result of [1] (where s_1, s_2 are easy to compute):

Corollary 2 *If* $m = 2$, $\sigma_i := \frac{d_j}{\gcd(d_1, d_2)}$ *with* $\{i, j\} = \{1, 2\}$ *and* $N := \sigma_1 d_1 - \nabla = \sigma_2 d_2 - \nabla$ *then the following minoration holds, for* $i = 1, 2$,

$$\deg G \geq q_i \cdot N + r_i d_i$$

where $\deg_{f_i} G = q_i \sigma_i + r_i$ *is the euclidean division of* $\deg_{f_i} G$ *by* σ_i.

Proof It suffices obviously to prove it for $i = 2$.

If $s_2 = \infty$ then, by Corollary 1,

$$\deg G \geq d_2 \cdot \deg_{f_2} G = d_2(q_2\sigma_2 + r_2) = q_2(d_2\sigma_2) + r_2 d_2 \geq q_2 \cdot N + r_2 d_2.$$

We now assume that $s_2 < \infty$. Actually one could prove the equality[2] $\sigma_i = s_i$ for $i = 1, 2$ and apply Corollary 1 directly. However it suffices actually to remark that, in

[2] As in [1] using Zaks Lemma, it is however possible to show it easily and without this result.

the theorem and hence in Corollary 1 as well, one may replace s_i by any smaller number. We are now left to prove, that $s_2 \geq \sigma_2$: s_2 is the degree of the minimal polynomial of $\bar{f_2}$ over $\mathrm{Frac}(\mathrm{gr}(K[f_1])) = \mathrm{Frac}(K[\bar{f_1}]) = K(\bar{f_1})$:

$$p(\bar{f_2}) = \bar{f}_2^{s_2} + p_{s_2-1}(\bar{f_1})\bar{f}_2^{s_2-1} + \cdots + p_1(\bar{f_1})\bar{f_2} + p_0(\bar{f_1}) = 0 \tag{5}$$

hence $\exists 0 \leq i \neq j \leq s_2$ such that $\deg p_i(\bar{f_1})\bar{f}_2^i = \deg p_j(\bar{f_1})\bar{f}_2^j$. It follows that $id_2 \equiv jd_2 \bmod d_1$ whence $d_1 \mid (i-j)d_2$ and $i - j \in \mathbb{Z}^* \frac{d_1}{\gcd(d_1,d_2)}$ which gives $s_2 \geq |i-j| \geq \frac{d_1}{\gcd(d_1,d_2)} = \sigma_2$.

Corollary 3 *Let G be a polynomial in $K[f_1, \ldots, f_m]$ such that* $\deg G = 1$. *Then, $\forall i = 1, \ldots, m$,* $\deg_{f_i} G = 0$ *or* $d_i = 1$ *or* $N_i = s_i d_i - \nabla \leq 1$.

Proof Otherwise, by Corollary 1, $\deg G = 1 \geq q_i N_i + r_i d_i \geq \min\{N_i, d_i\} \geq 2$, a contradiction.

Corollary 4 *Assume $m = n$ and $K[f_1, \ldots, f_n] = K[x_1, \ldots, x_n]$ i.e. $f_1, \ldots, f_n$ define an automorphism (well-known fact). Then $\forall i = 1, \ldots, n$, $d_i = 1$ or $s_i d_i \leq d_1 + \cdots + d_n - n + 1$. In particular, if $d_{\max} \geq d_j, \forall j$, and $d_{\max} \geq 2$ i.e. the automorphism is not affine, then $s_{\max} \leq n-1$ ($s_{\max}$ is the 's' corresponding to $d_{\max}$, not the biggest 's').*

Proof One has $\nabla = \nabla(f_1, \ldots, f_n) = d_1 + \cdots + d_n - n - \deg \mathrm{j}_{x_1,\ldots,x_n}(f_1, \ldots, f_n) = d_1 + \cdots + d_n - n$ (it is indeed well-known that the Jacobian of an automorphism lies in $K\backslash\{0\}$). Moreover $\forall j = 1, \ldots, n$, there exists $G_j \in K[f_1, \ldots, f_n]$ such that $x_j = G_j$ and, $\forall i = 1, \ldots, n$, $\deg_{f_i} G_j \geq 1$ for at least one $j = 1, \ldots, n$ otherwise $K[f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m] = K[x_1, \ldots, x_n]$ which is impossible. Whence, by Corollary 3, $d_i = 1$ or $s_i d_i \leq \nabla + 1 = d_1 + \cdots + d_n - n + 1$. With $d_{\max}$ one gets $s_{\max} d_{\max} \leq d_1 + \cdots + d_n - n + 1 \leq n d_{\max} - n + 1 \leq n d_{\max} - 1$ ($n \geq 2$) and it follows that $s_{\max} \leq n - 1$.

Corollary 5 (Tameness Theorem in dimension two) *Every automorphism of $K[x_1, x_2]$ is tame i.e. a product of affine and elementary ones. Recall that an automorphism $\tau : K[x_1, x_2] \to K[x_1, x_2]$ is called elementary when, up to exchanging x_1 and x_2, $\tau(x_1) = x_1 + p(x_2)$ and $\tau(x_2) = x_2$ for some $p(X) \in K[X]$.*

Proof Let $\alpha : K[x_1, x_2] \to K[x_1, x_2]$ be an automorphism defined by $\alpha(x_i) = f_i$ for $i = 1, 2$. We prove the corollary by induction on $d_1 + d_2 = \deg f_1 + \deg f_2$.

If $d_1 + d_2 = 2$ then $d_1 = d_2 = 1$ and α is affine.

Assume $d_1 + d_2 \geq 3$. Without loss of generality $d_1 \leq d_2$ and $d_2 \geq 2$ whence, by Corollary 4, $s_2 = 1$ and the relation (5) in the proof of Corollary 2 becomes: $\bar{f_2} = p(\bar{f_1})$ where $p(X)$ must be of the form $p(X) = p_{s_1} X^{s_1} \in K[X]$. Taking the elementary automorphism τ defined by $\tau(x_1) = x_1$ and $\tau(x_2) = x_2 - p(x_1)$ one has a new pair $f_1' := \alpha\tau(x_1) = \alpha(x_1) = f_1$ and $f_2' := \alpha\tau(x_2) = \alpha(x_2 - p(x_2)) = f_2 - p(f_1)$ with degrees $d_1' = d_1$ and $d_2' < d_2$ hence $d_1' + d_2' < d_1 + d_2$. By induction $\alpha\tau$ is tame and so is α.

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